

A class of nonlinear wave equations containing the continuous Toda case

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1998 J. Phys. A: Math. Gen. 31 2173

(<http://iopscience.iop.org/0305-4470/31/9/008>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.104

The article was downloaded on 02/06/2010 at 07:23

Please note that [terms and conditions apply](#).

A class of nonlinear wave equations containing the continuous Toda case

E Alfinito[†], M S Causo[‡], G Profilo[§] and G Soliani[‡]

[†] Dipartimento di Fisica dell'Università, 84084 Salerno, Italy, and Istituto Nazionale di Fisica Nucleare, Gruppo Collegato di Salerno, Italy

[‡] Dipartimento di Fisica dell'Università, 73100 Lecce, Italy, and Istituto Nazionale di Fisica Nucleare, Sezione di Lecce, Italy

[§] Dipartimento di Matematica dell'Università, 73100 Lecce, Italy

Received 11 February 1997, in final form 17 November 1997

Abstract. We consider a nonlinear field equation which can be derived from a binomial lattice as a continuous limit. This equation, containing a perturbative friction-like term and a free parameter γ , reproduces the Toda case (in the absence of the friction-like term) and other equations of physical interest, by choosing particular values of γ . We apply the symmetry and the approximate-symmetry approach, and the prolongation technique. Our main aim is to check the limits of validity of different analytical methods in the study of nonlinear field equations. We show that the equation under investigation with the friction-like term is characterized by a finite-dimensional Lie algebra admitting a realization in terms of boson annihilation and creation operators. In the absence of the friction-like term, the equation is linearized and connected with Bessel-type equations. Examples of exact solutions are displayed, and the algebraic structure of the equation is discussed.

0. Introduction

In this paper we study the class of nonlinear wave equations

$$u_{tt} + \epsilon u_t = \left[\left(1 + \frac{u}{\gamma} \right)^{\gamma-1} \right]_{xx} \quad (1)$$

where $u = u(x, t)$, subscripts denote partial derivatives, ϵ is a free parameter, and γ is a constant which takes all (real) values except zero and one.

For $\epsilon = 0$, equation (1) can be considered as the continuous limit of a uniform one-dimensional nonlinear lattice of N particles interacting through the nearest-neighbour potential [1]

$$\phi(r_n) = \frac{a_n}{b_n} \left[\left(1 + \frac{b_n r_n}{\gamma} \right)^\gamma - (1 + b_n r_n) \right] \quad (2)$$

where a_n and b_n are constants of the n th nonlinear spring.

The equation of motion of the chain reads

$$M_n \ddot{y}_n = -[\phi'(r_{n+1}) - \phi'(r_n)] \quad (3)$$

where $\phi' = \frac{d\phi}{dr}$, M_n is the mass of the n -particle and $r_n = y_n - y_{n+1}$, y_n is the displacement of the n -particle from its equilibrium position. We shall call the chain governed by the

potential (2) a ‘binomial lattice’. If the binomial lattice is uniform, i.e. $a_n = a$, $b_n = b$ and $M_n = M$ for any n , the equation of motion (3) takes the form

$$\ddot{r}_n = \frac{a}{M} \left[\left(1 + \frac{b}{\gamma} r_{n+2} \right)^{\gamma-1} - 2 \left(1 + \frac{b}{\gamma} r_{n+1} \right)^{\gamma-1} + \left(1 + \frac{b}{\gamma} r_n \right)^{\gamma-1} \right]. \quad (4)$$

For $\gamma \rightarrow \infty$, $\gamma = 2$, $\gamma = 3$ and $\gamma = -1$, equation (4) reproduces the Toda [2], the harmonic, the Fermi–Pasta–Ulam [3] and a Coulomb-like uniform lattice, respectively. Since equation (4) models interesting physical situations, its analysis may be important mostly from the point of view of a unifying theory of lattice systems described by the potential (2).

This difficult task is made simpler by starting from the investigation of the continuous limit of equation (4). Then, equation (4) becomes

$$u_{tt} = \left[\left(1 + \frac{u}{\gamma} \right)^{\gamma-1} \right]_{xx} \quad (5)$$

after rescaling $br \rightarrow u$ and $\sqrt{\frac{ab}{M}}t \rightarrow t$.

For $\gamma \rightarrow \infty$, equation (5) coincides with the $(1+1)$ -dimensional continuous Toda equation [4]

$$u_{tt} = (e^u)_{xx} \quad (6)$$

which is known also as *heavenly equation* and plays a crucial role in general relativity [5, 6].

Note that equation (5) is an example of hydrodynamical type [7]. This can be seen by putting $P = (1 + \frac{u}{\gamma})^{\gamma-1}$. Then, equation (5) may be written as

$$P_t = \frac{\gamma-1}{\gamma} P^{\frac{\gamma-2}{\gamma-1}} S_x \quad (7)$$

$$S_t = P_x. \quad (8)$$

(Equation (5) arises by eliminating S .)

Equation (1) represents a modified version of equation (5). It contains a friction-like term which can be considered as a small perturbation. This extension of equation (5) allows us to check the validity of different analytical methods usually employed to handle integrable equations.

Precisely, here we apply to equation (1) the symmetry [8] and the approximate-symmetry approach [9], and the prolongation procedure [10].

The main results achieved in this paper are the following. In section 2 we find the approximate symmetry generators of equation (1) in the case in which ϵ is a perturbative parameter. The commutation relations among these operators do not define an *exact* finite-dimensional algebra. (Exact algebras arise only for $\epsilon = 0$ and when $\epsilon \neq 0$ is not considered as a small quantity.) Notwithstanding, an exact finite-dimensional Lie algebra can be constructed by introducing certain auxiliary operators. A realization of this algebra, which characterizes the approximate symmetries of equation (1), is obtained in terms of boson annihilation and creation operators.

In section 3, the symmetry approach is exploited to determine special solutions of equation (1) and equation (5). In this context, the study of the symmetry corresponding to the generator X_4^0 (see (13)) is of particular interest. In fact, X_4^0 is a scale symmetry in the variables x, u together with a translation of $\frac{2\gamma}{\gamma-2}$ along u . For $\gamma \rightarrow \infty$, i.e. when equation (1) coincides with the continuous Toda equation, the scale symmetry is broken.

The reduced equation related to X_4^0 resembles the one-dimensional Liouville equation (see (38)) containing a square derivative which vanishes in the limit $\gamma \rightarrow \infty$. In the case $\epsilon = 0$, equation (38) is exactly solved for some values of the parameter γ . For example, if $\gamma = 3$ (a choice associated with the Fermi–Pasta–Ulam potential) we obtain a solution to equation (5) via equation (38) expressed in terms of the Weierstrass function.

For $\epsilon \neq 0$, we provide an implicit solution to equation (1) by using the symmetry variable $V = v\partial_x + \partial_t$, where v is a constant. This solution (see (76)) involves a generalization of the exponential-integral function. It is notable that for $\gamma \rightarrow \infty$, this solution reproduces just that corresponding to the continuous Toda case [11].

Section 4 is devoted to a study of equation (1) within the prolongation scheme [10]. This method is particularly convenient because it allows us to analyse equation (1) for $\epsilon = 0$ and $\epsilon \neq 0$ by an algebraic point of view. We show that the (differential) prolongation equations for equation (5) provide a class of solutions connected with the Lie algebra of the Euclidean group E_2 in the plane. This algebra leads to a linearized version of equation (5), in the sense that a mapping is established between equation (5) and the linear wave equation (85) written in a pseudopotential variable. For $\gamma \rightarrow \infty$, equation (85) coincides with a two-dimensional form of a linear wave equation that occurs in a generalized Gibbson–Hawking ansatz [12]. On the other hand, the prolongation equations for equation (1) can be solved in terms of power-series expansions whose coefficients (in the variable $z = (1 + \frac{u}{\gamma})^{\gamma-1}$) depend on the pseudopotential components and, for a certain infinite set of values of γ , obey a finite-dimensional Lie algebra. A representation of this algebra indicates the existence of a possible link between the prolongation method and the symmetry approach. Finally, in section 5, some comments are reported.

1. The approximate-symmetry approach

In the case in which ϵ is a perturbative parameter, equation (1) can be handled profitably by means of the method devised by Baikov *et al* [9] some years ago. This method enables one to construct approximate symmetries which are stable for small perturbations of the differential equation under investigation.

In order to obtain the approximate symmetries of equation (1) ($\epsilon \neq 0$), we have to introduce the quantities

$$\begin{aligned} F_0 &= u_{tt} - [(1 + u/\gamma)^{\gamma-1}]_{xx} \\ F_1 &= u_t \end{aligned} \tag{9}$$

into the equations (3.16) and (3.17) of [9].

For brevity, we shall omit any calculation and report below the results only.

The approximate-symmetry generator turns out to be

$$\begin{aligned} X &= \left[c_1 t + c_2 + \epsilon \left(\frac{\gamma - 2}{\gamma + 2} c_1 \frac{t^2}{2} + k_1 t + k_2 \right) \right] \partial_t \\ &\quad + [(c_1 + c_2)x + c_4 + \epsilon((k_1 + k_3)x + k_4)] \partial_x \\ &\quad + \left[\frac{2\gamma}{\gamma - 2} c_3 \left(1 + \frac{u}{\gamma} \right) + 2\epsilon \frac{\gamma}{\gamma - 2} \left(k_3 - \frac{\gamma - 2}{\gamma + 2} c_1 t \right) \left(1 + \frac{u}{\gamma} \right) \right] \partial_u \end{aligned} \tag{10}$$

with c_1, c_2, \dots and k_1, k_2, \dots arbitrary constants.

From (10) we obtain

$$X_1 = X_1^0 + \epsilon \left[\frac{\gamma - 2}{\gamma + 2} \frac{t^2}{2} \partial_t - \frac{2\gamma t}{\gamma + 2} \left(1 + \frac{u}{\gamma} \right) \partial_u \right] \tag{11}$$

where

$$X_1^0 = t\partial_t + x\partial_x \quad (12)$$

$$X_2 \equiv X_2^0 = \partial_t$$

$$X_3 \equiv X_3^0 = \partial_x$$

$$X_4 \equiv X_4^0 = x\partial_x + \frac{2\gamma}{\gamma-2} \left(1 + \frac{u}{\gamma}\right) \partial_u$$

$$X_5 = \epsilon(t\partial_t + x\partial_x) \quad (13)$$

$$X_6 = \epsilon\partial_t$$

$$X_7 = \epsilon \left[x\partial_x + \frac{2\gamma}{\gamma-2} \left(1 + \frac{u}{\gamma}\right) \partial_u \right]$$

$$X_8 = \epsilon\partial_x.$$

The operators $X_1^0, X_2^0, X_3^0, X_4^0$ are the (exact) symmetry generators of equation (1) for $\epsilon = 0$, while X_1, X_2, X_3, X_4 are the *approximate* symmetry generators of equation (1) for $\epsilon \neq 0$. The *exact* symmetry generators of equation (1) for $\epsilon \neq 0$ are X_1^0, X_2^0 and X_3^0 . The operators $X_5^0, X_6^0, X_7^0, X_8^0$ are inessential.

The approximate symmetry generators X_1, X_2, X_3, X_4 satisfy the commutation relations

$$[X_2, X_1] = X_2 + \epsilon \left[\frac{\gamma-2}{\gamma+2} t\partial_t - \frac{2\gamma}{\gamma+2} \left(1 + \frac{u}{\gamma}\right) \partial_u \right] \quad (14)$$

$$[X_2, X_4] = 0 \quad [X_2, X_3] = 0 \quad [X_1, X_4] = 0 \quad (15)$$

$$[X_1, X_3] = -X_3 \quad [X_3, X_4] = X_3. \quad (16)$$

The commutation rules (14)–(16) define an exact finite-dimensional Lie algebra only for $\epsilon = 0$. However, they can be exploited to build up a realization of an exact Lie algebra, which holds for $\epsilon \neq 0$, with the help of the ‘auxiliary’ operators

$$Z = \frac{\gamma-2}{\gamma+2} t\partial_t - \frac{2\gamma}{\gamma+2} \left(1 + \frac{u}{\gamma}\right) \partial_u \quad (17a)$$

$$Y = \frac{\gamma-2}{\gamma+2} t^2 \partial_t - \frac{2\gamma t}{\gamma+2} \left(1 + \frac{u}{\gamma}\right) \partial_u. \quad (17b)$$

In doing so, we find

$$[X_2, X_1] = X_2 + \epsilon Z \quad (18a)$$

$$[X_2, X_4] = [X_2, X_3] = [X_1, X_4] = 0 \quad (18b)$$

$$[X_1, X_3] = -X_3 \quad [X_3, X_4] = X_3 \quad (18c)$$

$$[X_1, Z] = -\epsilon \frac{\gamma-2}{\gamma+2} Y \quad [X_1, Y] = Y \quad (18d)$$

$$[X_2, Y] = Z \quad [X_2, Z] = \frac{\gamma-2}{\gamma+2} X_2 \quad [Z, Y] = \frac{\gamma-2}{\gamma+2} Y \quad (18e)$$

$$[X_3, Y] = [X_3, Z] = [X_4, Y] = [X_4, Z] = 0. \quad (18f)$$

Hereafter, the symbols Y, Z, X_j ($j = 1, \dots, 4$) will mean both the abstract elements and the corresponding realizations (see (13), (17) and (18)) of the finite-dimensional Lie algebra (18).

At this stage, we remark that the subalgebra (18e) is isomorphic to the $sl(2, R)$ algebra

$$[Z', T] = 2S \quad [T, S] = 2Z' \quad [S, Z'] = -2T \quad (19)$$

where

$$T = \sqrt{2\frac{\gamma+2}{\gamma-2}}(Y + X_2) \quad S = \sqrt{2\frac{\gamma+2}{\gamma-2}}(Y - X_2) \quad (20)$$

$$Z' = 2\frac{\gamma+2}{\gamma-2}Z.$$

It is straightforward to prove the following proposition.

Proposition 1. The Casimir operator

$$C = T^2 - S^2 - Z'^2$$

$$\equiv 4\frac{\gamma+2}{\gamma-2} \left[2X_2Y - Z \left(\frac{\gamma+2}{\gamma-2}Z + 1 \right) \right] \quad (21)$$

of the Lie algebra (19), commutes with all the elements Y, Z, X_j ($j = 1, \dots, 4$) of the Lie algebra (18).

Furthermore, the following proposition holds.

Proposition 2. The Lie algebra (18) admits a realization in terms of boson annihilation and creation operators.

This can be seen by setting

$$a_1^+ = t \quad a_2^+ = u \quad a_3^+ = x \quad a_1 = \partial_t \quad a_2 = \partial_u \quad a_3 = \partial_x \quad (22)$$

to give

$$[a_j, a_k^+] = \delta_{jk} \quad [a_j, a_k] = 0 \quad [a_j^+, a_k^+] = 0 \quad (j, k = 1, 2, 3) \quad (23)$$

$$Y = \frac{\gamma-2}{2(\gamma+2)} a_1^{+2} a_1 - \frac{2\gamma a_1^+}{\gamma+2} \left(1 + \frac{1}{\gamma} a_2^+ \right) a_2$$

$$Z = \frac{\gamma-2}{\gamma+2} a_1^+ a_1 - \frac{2\gamma}{\gamma+2} \left(1 + \frac{1}{\gamma} a_2^+ \right) a_2 \quad (24)$$

$$X_1 = a_1^+ a_1 + a_3^+ a_3 + \epsilon Y \quad X_2 = a_1 \quad X_3 = a_2$$

$$X_4 = a_3^+ a_3 + \frac{2\gamma}{\gamma-2} \left(1 + \frac{1}{\gamma} a_2^+ \right) a_2$$

and

$$C = \left(\frac{\gamma}{\gamma-2} \right)^2 C_\infty + C(\gamma) \quad (25)$$

where $C_\infty = -8a_2(2a_2 + 1)$ is the Casimir invariant relative to the continuous Toda case $\gamma \rightarrow \infty$, and the operator $C(\gamma)$, defined by

$$C(\gamma) = -\frac{8a_2^+ a_2}{\gamma-2} \left[1 + \frac{2\gamma}{\gamma-2} \left(2a_2 + \frac{1}{\gamma} a_2^+ a_2 \right) \right] \quad (26)$$

tends to zero as $\gamma \rightarrow \infty$.

Although the full role of the closed algebra (18) has to be better understood, it is noteworthy that the auxiliary operators Z and Y (see (17)) can be interpreted as symmetry variables of the equations

$$u_{xx} + \epsilon u_x - \frac{2}{\gamma} \left(1 + \frac{u}{\gamma} \right)^{-2} u_x^2 = - \left(1 + \frac{u}{\gamma} \right)^2 \left[\left(1 + \frac{u}{\gamma} \right)^{1-\gamma} \right]_{tt} \quad (27)$$

$$u_{xx} + \epsilon u_x - \frac{3}{2\gamma} \left(1 + \frac{u}{\gamma} \right)^{-1} u_x^2 = -2 \left(1 + \frac{u}{\gamma} \right)^{\frac{3}{2}} \left[\left(1 + \frac{u}{\gamma} \right)^{\frac{1-\gamma}{2}} \right]_{tt} \quad (28)$$

respectively.

Equations (27) and (28) arise formally from equation (1) via the transformations

$$x \rightarrow t \quad u \rightarrow -\frac{u}{1 + \frac{u}{\gamma}} \tag{29}$$

and

$$x \rightarrow t \quad u \rightarrow \gamma \left[\left(1 + \frac{u}{\gamma} \right)^{-1/2} - 1 \right]. \tag{30}$$

From Z we obtain the invariants $x' = x$, $\eta'(x') = \eta(x)$, with

$$\eta(x) = \left(1 + \frac{u}{\gamma} \right)^{\frac{\gamma-2}{2}} t \tag{31}$$

which gives the reduced equation

$$\eta_x^2 + \frac{2-\gamma}{\gamma} \eta \eta_{xx} + \frac{2-\gamma}{\gamma} \epsilon \eta \eta_x = \frac{\gamma-1}{\gamma} \tag{32}$$

from substitution into (27). Note that equation (32) coincides, formally, with equation (37) for $x \rightarrow t$ (see section 3). Therefore, putting $\eta = e^W$, equation (32) becomes equation (38) where t is replaced by x .

On the other hand, the operator Y yields the invariants $x' = x$, $r'(x') = r(x)$, with

$$r(x) = \left(1 + \frac{u}{\gamma} \right)^{\frac{\gamma-2}{2}} t \tag{33}$$

which leads to the ordinary differential equation of the modified Liouville type (see (38))

$$v_{xx} + \epsilon v_x + \frac{1}{\gamma-2} v_x^2 = \frac{2}{\gamma-2} \left[\frac{1}{\gamma} - 2(1+\gamma) \right] e^{-2v} \tag{34}$$

where $v = \ln r$.

We point out that in the continuous Toda case, i.e. for $\gamma \rightarrow \infty$, in the reduced equations of the one-dimensional modified Liouville type ((38), written in terms of the variable x and (34)) the terms W_t^2 , W_x^2 and v_x^2 disappear. This is due to the fact that for $\gamma \rightarrow \infty$, the coefficient in front of $u \partial_u$ in X_4^0 is vanishing, so that X_4^0 is no longer a scale symmetry in the variables x, u (together with a translation of $\frac{2\gamma}{\gamma-2}$ along u). Thus, the presence of square first derivatives in the reduced equations generated by X_4^0 reflects the existence of a scale symmetry, which is broken in the limit $\gamma \rightarrow \infty$.

2. Explicit solutions

Here we shall display some significant examples of exact solutions to equation (1) by using the symmetry approach. To this aim, first let us deal with the generator X_4^0 which appears in both the cases $\epsilon = 0$ and $\epsilon \neq 0$. The group transformations involved by X_4^0 are

$$x' = e^\lambda x \tag{35a}$$

$$t' = t \tag{35b}$$

$$u' = \gamma \left[\left(1 + \frac{u}{\gamma} \right) e^{\frac{2\lambda}{\gamma-2}} - 1 \right] \tag{35c}$$

where λ is the group parameter. A set of basis invariants corresponding to (35) is given by

$$I_1 = \left(1 + \frac{u'}{\gamma}\right)^{\frac{2-\gamma}{2}} x' = \left(1 + \frac{u}{\gamma}\right)^{\frac{2-\gamma}{2}} x \tag{36a}$$

$$I_2 = t' = t. \tag{36b}$$

Now, by putting $\rho(t) = I_1$, from equation (1) we obtain the reduced equation

$$\rho_t^2 + \frac{2-\gamma}{\gamma} \rho \rho_{tt} + \frac{2-\gamma}{\gamma} \epsilon \rho \rho_t = \frac{\gamma-1}{\gamma}. \tag{37}$$

Through the change of variable $\rho = e^W$, equation (37) becomes

$$W_{tt} + \epsilon W_t + \frac{2}{2-\gamma} W_t^2 = \frac{\gamma-1}{2-\gamma} e^{-2W} \tag{38}$$

which is a kind of modified one-dimensional Liouville equation. For $\epsilon = 0$, equation (38) can be solved exactly. This occurs via the position

$$W = \frac{2-\gamma}{2} \ln \theta \tag{39}$$

which transforms equation (38) into

$$\theta_{tt} = \frac{2(\gamma-1)}{(2-\gamma)^2} \theta^{\gamma-1}. \tag{40}$$

From (40) one easily finds

$$\theta_t^2 = \frac{4(\gamma-1)}{(2-\gamma)^2 \gamma} \theta^\gamma + c \tag{41}$$

where c is a constant of integration.

Putting

$$y^2 = 1 + a\theta^\gamma \tag{42}$$

with

$$a = \frac{4(\gamma-1)}{c\gamma(\gamma-2)^2} \tag{43}$$

equation (41) provides

$$\int dy (y^2 - 1)^{\frac{1-\gamma}{\gamma}} = \frac{\gamma}{2} a^{\frac{1}{\gamma}} \sqrt{ct} + \text{constant} \tag{44}$$

with $c > 0$.

In some situations (see case (iv)) it may be convenient to write formula (44) as

$$\int (\cosh z)^{-\frac{2}{\gamma}} dz = (-1)^{\frac{\gamma}{1-\gamma}} \frac{\gamma}{2} a^{\frac{1}{\gamma}} \sqrt{ct} + \text{constant} \tag{45}$$

which follows from (44) via the change of variable $y = \tanh z$.

Once the integral at the left-hand side of (44) has been calculated, one obtains the *exact* solution

$$u = \gamma \left[x^{\frac{2}{\gamma-2}} \left(\frac{y^2 - 1}{a} \right)^{\frac{1}{\gamma}} - 1 \right] \tag{46}$$

to equation (1) ($\epsilon = 0$) (see (36a), (39), and (42)).

At this point, by way of example, we would like to deal with some particular value of the parameter γ which is compatible with an *explicit* expression of y in terms of t .

In what follows we shall deal with the cases (i) $\gamma = \frac{1}{2}$, (ii) $\gamma = 3$, and (iii) $\gamma = -2$.

The last two choices correspond, respectively, to the Fermi–Pasta–Ulam potential [3], and to a potential whose nonlinear part, $1/(1 - \frac{u}{2})^2$, mimics the inverse square potential appearing in the treatment of the scattering states in conformally invariant quantum mechanics [13].

Case 1. For $\gamma = \frac{1}{2}$, from (44) we have

$$\int (y^2 - 1) dy = \frac{1}{4}a^2\sqrt{ct} \quad (47)$$

with $a = -\frac{16}{9c}$, where the constant at the right-hand side of (44) has been put equal to zero.

Equation (47) yields the cubic equation

$$y^3 - 3y - \frac{3}{4}a^2\sqrt{ct} = 0 \quad (48)$$

which provides the real solution [14, p 17]

$$y = s_1 + s_2 \quad (49)$$

where

$$s_1 = \left(\frac{3}{8}a^2\sqrt{ct} + \sqrt{\frac{9}{64}a^4ct^2 - 1} \right)^{\frac{1}{3}} \quad (50a)$$

$$s_2 = \left(\frac{3}{8}a^2\sqrt{ct} - \sqrt{\frac{9}{64}a^4ct^2 - 1} \right)^{\frac{1}{3}} \quad (50b)$$

for $t > \frac{8}{3a^2\sqrt{c}}$.

The remaining solutions of (48) are complex conjugate functions, and are given by

$$y_1 = -\frac{1}{2}(s_1 + s_2) + i\frac{\sqrt{3}}{2}(s_1 - s_2) \quad y_2 = y_1^* \quad (51)$$

Now, by inserting (51) in (48) for $\gamma = \frac{1}{2}$ we find the exact solution

$$u = \frac{1}{2} \left[\frac{1}{a^2}x^{-\frac{4}{3}}(s_1^2 + s_2^2 + 1) - 1 \right]. \quad (52)$$

Case 2. For $\gamma = 3$, equation (41) provides

$$\theta_t^2 = \frac{8}{3}\theta^3 + c \quad (53)$$

which can be written as

$$\theta_\tau^2 = 4\theta^3 - g_3 \quad (54)$$

via the rescaling $\tau = \sqrt{\frac{2}{3}}t$, with $g_3 = -c/2$. Equation (54) is a special version of the equation

$$\theta_\tau^2 = 4\theta^3 - g_2\theta - g_3 \quad (55)$$

which is satisfied by the Weierstrass elliptic function $\mathcal{P}(\tau; g_2, g_3)$, where g_2 and g_3 are the invariants of \mathcal{P} (see [14, p 918]).

Concerning equation (54) we shall distinguish two cases, $g_3 = 0$ and $g_3 \neq 0$. For $g_3 = 0$, we have

$$\theta(\tau) = \mathcal{P}(\tau; 0, 0) = \frac{1}{\tau^2} \quad (56)$$

which coincides with the first term of the series representation of $\mathcal{P}(\tau; g_2, g_3)$.

Then, keeping in mind (42) and (48), we find

$$u = 3 \left(\frac{3x^2}{2t^2} - 1 \right). \tag{57}$$

For $g_3 \neq 0$, equation (54) gives

$$\int_{\infty}^{\theta} \frac{d\alpha}{\sqrt{4\alpha^3 - g_3}} = \tau \tag{58}$$

that is to say

$$\theta = \mathcal{P}(\tau; 0, g_3). \tag{59}$$

Hence, from (46) we deduce

$$u = 3(x^2\theta - 1) = 3[x^2\mathcal{P}(\tau; 0, g_3) - 1]. \tag{60}$$

For $c = -2$ ($g_3 = 1$), the Weierstrass function in (60) reduces to the equianharmonic case $\mathcal{P}(\tau; 0, 1)$ [14, p 652].

Finally, note that another case which can be solved in terms of Weierstrass function corresponds to the choice $\gamma = 4$.

Case 3. If $\gamma = -2$, from (44) we easily obtain

$$u = -2 \left(\sqrt{\frac{ct^2 - a}{x}} - 1 \right) \tag{61}$$

with $a = \frac{3}{8c}$.

Case 4. For $\gamma = -1$, the potential (2) takes the form $\phi \sim (1 - u)^{-1} - (1 + u)$, which resembles a special case of the generalized Killingbeck potential $V = -\frac{A}{r} + Br + Cr^2$ [15]. We remind the reader that a Coulomb potential perturbed by a linear term describes the spherical Stark effect in hydrogen [16]. Putting in (45) $\gamma = -1$ and $c = -|c|$, we arrive at the relation

$$2z + \sinh 2z = \frac{9}{4}|c|^{\frac{3}{2}}t + \text{constant}. \tag{62}$$

Curiously enough, by replacing formally the hyperbolic sine by the exponential, equation (62) becomes an equation of the Schrödinger type, which appears in the bootstrap model and in renormalization theory, whose analytical structure was investigated by Hagedorn and Rafelsky [17]. These authors obtained a solution of the Schrödinger equation both as a power series expansion and as an integral representation. Thus, it should be of interest to try to adopt the same strategy in the study of equation (62).

Another nontrivial example of exact solution to equation (1) can be determined starting from the linear combination of $X_3 \equiv X_3^0 = \partial_x$ and $X_2 \equiv X_2^0 = \partial_t$:

$$V = v\partial_x + \partial_t \tag{63}$$

where v is a (real) constant. The group transformations are $x' = x + v\lambda$, $t' = t + \lambda$, which provide the invariant $\xi = x' - vt' = x - vt$. Thus, by inserting the variable ξ into equation (1), we obtain the ordinary differential equation

$$v^2 u_{\xi\xi} + \epsilon v u_{\xi} = \left[\left(1 + \frac{u}{\gamma} \right)^{\gamma-1} \right]_{\xi\xi} \tag{64}$$

which gives

$$v^2 u_\xi - \epsilon v u = \left[\left(1 + \frac{u}{\gamma} \right)^{\gamma-1} \right]_\xi + c_0 \tag{65}$$

where c_0 is a constant of integration. By choosing $c_0 = 0$, $v^2 = \frac{\gamma-1}{\gamma}$ and limiting ourselves to consider those values of γ such that $\gamma = -|\gamma|$, equation (65) yields

$$\left[1 - \left(1 - \frac{u}{|\gamma|} \right)^{-|\gamma|-2} \right]_\xi = \frac{\epsilon}{v} u \tag{66}$$

from which

$$\int_0^u \frac{1}{u'} \left[1 - \left(1 + \frac{u'}{|\gamma|} \right)^{-|\gamma|-2} \right] du' = \frac{\epsilon}{v} (\xi - \xi_0) \tag{67}$$

ξ_0 being a constant of integration.

Note that for $|\gamma| \rightarrow \infty$, equation (67) becomes

$$\int_0^u \frac{1 - e^{-u'}}{u'} du' \equiv Ein(u) = \frac{\epsilon}{v} (\xi - \xi_0) \tag{68}$$

where $Ein(u)$ denotes the exponential-integral function [18, p 255]

$$Ein(u) = -Ei(-u) + \ln u + C \tag{69}$$

$Ei(-u) = -\int_u^\infty \frac{e^{-u'}}{u'} du'$, and C is the Euler–Mascheroni constant, defined by [18]

$$C = -\psi(1) = -\int_0^\infty e^{-t} \ln t dt \tag{70}$$

where $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ is the psi (or digamma) function [14, p 258].

At this point, let us introduce the function

$$\begin{aligned} \hat{\Gamma}(\alpha, u; |\gamma|) &= \int_u^\infty \left(1 + \frac{u'}{|\gamma|} \right)^{-(2+|\gamma|)} (u')^{\alpha-1} du' \\ &= \hat{\Gamma}(\alpha, 0; |\gamma|) - \int_0^u \left(1 + \frac{u'}{|\gamma|} \right)^{-(2+|\gamma|)} (u')^{\alpha-1} du' \end{aligned} \tag{71}$$

which holds for $0 < \alpha < 2 + |\gamma|$.

We point out that for $|\gamma| \rightarrow \infty$, $\hat{\Gamma}(\alpha, u; |\gamma|)$ reproduces the incomplete gamma function $\Gamma(\alpha, x)$ [14, p 260].

Now, since the integral at the right-hand side of (71) can be written as [18]

$$\begin{aligned} \int_0^u (u')^{\alpha-1} \left(1 + \frac{u'}{|\gamma|} \right)^{-(2+|\gamma|)} du' &= \frac{u^\alpha}{\alpha} {}_2F_1 \left(2 + |\gamma|, \alpha; \alpha + 1, -\frac{u}{|\gamma|} \right) \\ &= \frac{u^\alpha}{\Gamma(2 + |\gamma|)} \sum_{n=0}^\infty \frac{\Gamma(2 + |\gamma| + n)}{(\alpha + n)n!} (-1)^n \left(\frac{u'}{|\gamma|} \right)^n. \end{aligned} \tag{72}$$

Equation (71) takes the form

$$\hat{\Gamma}(\alpha, u; |\gamma|) = \hat{\Gamma}(\alpha, 0; |\gamma|) - \frac{u^\alpha}{\alpha} - \frac{u^\alpha}{\Gamma(2 + |\gamma|)} \sum_{n=1}^\infty \frac{\Gamma(2 + |\gamma| + n)}{(\alpha + n)n!} (-1)^n \left(\frac{u}{|\gamma|} \right)^n \tag{73}$$

where $\Gamma(\cdot)$ is the gamma-function and ${}_2F_1$ is the Gauss hypergeometric series, respectively [14, p 556].

We have (see [18, p 209])

$$\begin{aligned} \lim_{\alpha \rightarrow 0} \hat{\Gamma}(\alpha, u; |\gamma|) &= \lim_{\alpha \rightarrow 0} \left[\frac{\alpha \hat{\Gamma}(\alpha, 0; |\gamma|) - u^\alpha}{\alpha} \right. \\ &\quad \left. - \frac{u^\alpha}{\Gamma(2 + |\gamma|)} \sum_{n=1}^{\infty} \frac{\Gamma(2 + |\gamma| + n)}{(\alpha + n)n!} (-1)^n \left(\frac{u}{|\gamma|}\right)^n \right] \\ &= -\ln u + \ln |\gamma| - \psi(2 + |\gamma|) - C - \int_0^u \frac{1}{u'} \left[\left(1 + \frac{u'}{|\gamma|}\right)^{-(2+|\gamma|)} - 1 \right] du'. \end{aligned} \tag{74}$$

On the other hand, from (71) we obtain (see [14, p 255])

$$\lim_{\alpha \rightarrow 0} \hat{\Gamma}(\alpha, u; |\gamma|) = \int_u^\infty \left(1 + \frac{u'}{|\gamma|}\right)^{-(2+|\gamma|)} \frac{1}{u'} du'. \tag{75}$$

Finally, by comparing (75) with (74), we find

$$\begin{aligned} \int_u^\infty \left(1 + \frac{u'}{|\gamma|}\right)^{-(2+|\gamma|)} \frac{1}{u'} du' &= \ln \frac{|\gamma|}{u} - \psi(2 + |\gamma|) - C \\ &\quad + \int_0^u \frac{1}{u'} \left[1 - \left(1 + \frac{u'}{|\gamma|}\right)^{-(2+|\gamma|)} \right] du' \end{aligned} \tag{76}$$

that is (see (67))

$$\begin{aligned} \hat{\Gamma}(0, u; |\gamma|) &= \int_u^\infty \left(1 + \frac{u'}{|\gamma|}\right)^{-(2+|\gamma|)} \frac{1}{u'} du' \\ &= \frac{\epsilon}{v}(\xi - \xi_0) + \ln \frac{|\gamma|}{u} - \psi(2 + |\gamma|) - C. \end{aligned} \tag{77}$$

We remark that $\lim_{|\gamma| \rightarrow \infty} [\ln |\gamma| - \psi(2 + |\gamma|)] = 0$ [19, p 945].

Therefore, equation (77) becomes

$$\lim_{|\gamma| \rightarrow \infty} \hat{\Gamma}(0, u; |\gamma|) = -Ei(-u) = \frac{\epsilon}{v}(\xi - \xi_0) - \ln u - C \tag{78}$$

from which (see (69))

$$Ein(u) = \frac{\epsilon}{v}(\xi - \xi_0). \tag{79}$$

This result, corresponding to the continuous Toda case, has been already obtained in [11]. Consequently, the function $\hat{\Gamma}(\alpha, u; |\gamma|)$ defined by (71) can be considered as an extended version of the incomplete gamma function $\Gamma(\alpha, u) = \int_u^\infty e^{-t} t^{\alpha-1} dt$ (see [14, p 260]).

The quantity (77), where $\hat{\Gamma}(0, u; |\gamma|)$ can be interpreted as a generalization of the exponential-integral function, constitutes an implicit solution of equation (1).

3. Linearization and algebraic properties

Equation (1) can be handled within the prolongation scheme [10]. In doing so, let us consider the prolongation equations for equation (1):

$$y_x^i = F^i(u, u_t; y) \quad y_t^j = G^j(u, u_{x_j}; y) \tag{80}$$

where $i = 1, 2, \dots, N$ (N arbitrary) and the set of variables $y \equiv \{y^i\}$ is the pseudopotential [10] ($j = 1, 2, \dots, M$ (M arbitrary)).

The compatibility condition for equations (80) gives

$$F^i = \frac{\gamma}{\gamma-1} L^i(y) u_t + M^i(u, y) \quad (81a)$$

$$G^i = L^i(y) \left(1 + \frac{u}{\gamma}\right)^{\gamma-2} u_x + P^i(u; y) \quad (81b)$$

where $M^i = M^i(u; y)$, $P^i = P^i(u; y)$ and $L^i = L^i(y)$ are defined by

$$\frac{\gamma-1}{\gamma} M_u^i + [P, L]^i = \epsilon L^i \quad (82a)$$

$$\left(1 + \frac{u}{\gamma}\right)^{\gamma-2} [L, M]^i = P_u^i \quad (82b)$$

$$[M, P]^i = 0 \quad (82c)$$

with $[P, L]^i = P^k \frac{\partial L^i}{\partial y^k} - L^k \frac{\partial P^i}{\partial y^k}$, and so on.

In order to explore the prolongation equations (82), for brevity we shall omit the index i .

The following proposition is noteworthy.

Proposition 3. Let u be a solution of equation (5). Then, the function $y_2 = y_2(x, t)$ defined by

$$y_{2x} = W(u) \sinh y_1 \quad y_{2t} = S(u) \cosh y_1 \quad (83)$$

$$y_{1x} = \frac{\gamma}{\gamma-1} u_t \quad y_{1t} = \left(1 + \frac{u}{\gamma}\right)^{\gamma-2} u_x \quad (84)$$

satisfies the wave equation

$$y_{2tt} = \frac{\gamma-1}{\gamma} \left(1 + \frac{u}{\gamma}\right)^{\gamma-2} y_{2xx} \quad (85)$$

where $W(u)$ and $S(u)$ obey the linear differential equations of the Bessel type

$$W_{uu} = \frac{\gamma}{\gamma-1} \left(1 + \frac{u}{\gamma}\right)^{\gamma-2} W \quad (86)$$

and

$$S_{uu} = \frac{\gamma}{\gamma-1} \left(1 + \frac{u}{\gamma}\right)^{\gamma-2} S + \frac{\gamma-2}{\gamma} \left(1 + \frac{u}{\gamma}\right)^{-1} S_u. \quad (87)$$

To prove this proposition, let us look for a solution to equations (82) of the form

$$M = W(u)V(y) \quad P = S(u)T(y). \quad (88)$$

Then equations (82) provide

$$S = \frac{\gamma-1}{\gamma} W_u \quad S_u = \left(1 + \frac{u}{\gamma}\right)^{\gamma-2} W \quad (89)$$

and

$$[X_1, X_2] = X_3 \quad [X_1, X_3] = X_2 \quad [X_2, X_3] = 0. \quad (90)$$

with $L \equiv X_1$, $V \equiv X_2$, $T \equiv X_3$.

Equations (89) imply equations (86) and (87). By means of the change of variable

$$z = 2i \sqrt{\frac{\gamma}{\gamma-1}} \left(1 + \frac{u}{\gamma}\right)^{\frac{\gamma}{2}}. \quad (91)$$

Equation (86) is transformed into the Bessel equation

$$z^2 Z_{zz} + z Z_z + \left(z^2 - \frac{1}{\gamma^2} \right) Z = 0 \tag{92}$$

where $Z = Z(z)$ is related to W by

$$W = \left(1 + \frac{u}{\gamma} \right)^{\frac{1}{2}} Z \left[2i \sqrt{\frac{\gamma}{\gamma-1}} \left(1 + \frac{u}{\gamma} \right)^{\frac{\gamma}{2}} \right]. \tag{93}$$

On the other hand, equation (87) takes the form

$$z^2 Z_{zz} + z Z_z + \left[z^2 - \left(\frac{\gamma-1}{\gamma} \right)^2 \right] Z = 0 \tag{94}$$

where

$$S = \left(1 + \frac{u}{\gamma} \right)^{\frac{\gamma-1}{2}} Z \left[2i \sqrt{\frac{\gamma}{\gamma-1}} \left(1 + \frac{u}{\gamma} \right)^{\frac{\gamma}{2}} \right]. \tag{95}$$

Z stands for a generic Bessel function of index $\pm \frac{1}{\gamma}$ and $\pm \frac{\gamma-1}{\gamma}$, respectively [14, p 358].

We observe that in the study of the heavenly equation, Bessel functions appear in different contexts. For example, in [20] Bessel functions of first and zero order turn out to be related to special solutions of Einstein field equations. Conversely, in [11] Bessel functions of the same order are involved in the pseudopotential formulation of the heavenly equation.

The commutation rules (90) define the Lie algebra corresponding to the Euclidean group E_2 in the plane. A realization of (90) in terms of a two-component pseudopotential $y \equiv (y_1, y_2)$ is

$$X_1 = \partial_{y_1} \quad X_2 = \sinh y_1 \partial_{y_2} \quad X_3 = \cosh y_1 \partial_{y_2}. \tag{96}$$

Therefore, with the help of (81a) and (81b), equations (80) take the form expressed by (83) and (84). Furthermore, by differentiating y_{2x} with respect to x and y_{2t} with respect to t (see (83)) and using (84) and (89), one arrives at the wave equation (85).

Finally, we observe that for $\gamma = 3$, i.e. in the case of the Fermi–Pasta–Ulam potential, equation (86) leads to the Airy equation [14, p 446]

$$W_{\sigma\sigma} = \sigma W \tag{97}$$

via the transformation $\sigma = 2^{-\frac{1}{3}}(u + 3)$.

At this stage, we remark that equation (85), which is linear in the pseudopotential variable y_2 , represents a correspondence between any solution u of the original equation (5) and y_2 . This feature allows us, in theory, to solve nontrivial linear wave equations using proposition 3. To provide an explicit example, let us consider the solution (57) related to the case $\gamma = 3$. Then, equation (85) becomes

$$y_{2tt} = \frac{x^2}{t^2} y_{2xx}. \tag{98}$$

Inserting (57) into equations (84) and (85), we find

$$y_{2x} = W(u) \sinh \left(c - \frac{g}{2} \frac{x^3}{t^3} \right) \tag{99}$$

$$y_{2t} = \frac{2}{3} W_u(u) \cosh \left(c - \frac{g}{2} \frac{x^3}{t^3} \right) \tag{100}$$

where $u = 2^{1/3}\sigma - 3$, $\sigma = 9 \times 2^{-\frac{4}{3}} \frac{x^2}{z^2}$, c is a constant of integration, and $W(\sigma)$ is the Airy function (see equation (97)).

The prolongation equations (82) offer the possibility of getting a further insight into the algebraic structure of equation (1).

To this aim, let us write equations (82) in the form

$$\begin{aligned} \frac{\gamma-1}{\gamma} P_z + [M, L] &= 0 & \frac{\gamma-1}{\gamma} z^{\frac{\gamma-2}{\gamma-1}} M_z &= \epsilon L + [L, P] \\ [M, P] &= 0 \end{aligned} \quad (101)$$

where $z = (1 + \frac{u}{\gamma})^{\gamma-1}$. Then, we look for a solution to equations (101) such that

$$M = \sum_{k=0}^{\infty} a_k(y) z^k \quad P = \sum_{k=0}^{\infty} b_k(y) z^k. \quad (102)$$

In the following, we limit ourselves to characterize mainly the algebraic structure of equation (1) for a particular sequel of values of the parameter γ (see proposition 4). In this case it turns out that a finite-dimensional Lie algebra is associated with equation (1). This algebra is used to write equation (1) in a potential form, which allows us to establish some analogies between the prolongation and the symmetry approaches. In general, namely for any value of γ (provided that $\gamma \neq 0, 1$) a systematic analysis of the algebraic properties of equation (1), which is based on the ansatz (102), requires further efforts.

Substitution from (102) into equation (101) gives the following constraints between the coefficients $a_k(y)$ and $b_k(y)$:

$$k \frac{\gamma-1}{\gamma} b_k + [a_{k-1}, L] = 0 \quad (103a)$$

$$\frac{\gamma-1}{\gamma} (a_1 + 2a_2z + 3a_3z^2 + 4a_4z^3 + \dots) z^{\frac{\gamma-2}{\gamma-1}} = \epsilon L + [L, b_0 + b_1z + b_2z^2 + b_3z^3 + \dots] \quad (103b)$$

and

$$\begin{aligned} [a_0, b_0] &= 0 \\ [a_0, b_1] + [a_1, b_0] &= 0 \\ [a_0, b_2] + [a_1, b_1] + [a_2, b_0] &= 0 \\ \dots & \\ \sum_{k=1}^N [a_{k-1}, b_{N-k}] &= 0 \end{aligned} \quad (104)$$

with N arbitrary.

The following property holds.

Proposition 4. If $\epsilon \neq 0$ and $\nu = \frac{\gamma-2}{\gamma-1}$ is such that $\nu \neq \dots, -3, -2, -1, 0, 1, 2, 3, \dots$, equations (104) imply the finite-dimensional Lie algebra \mathcal{L}

$$[a_0, b_0] = [a_0, b_1] = [L, b_1] = 0 \quad (105a)$$

$$[b_0, b_1] = \epsilon b_1 \quad (105b)$$

$$[b_0, L] = \epsilon L \quad (105c)$$

$$[a_0, L] = \frac{1}{\nu-2} b_1 \quad (105d)$$

for $\epsilon \neq 0$, and the finite-dimensional Lie algebra \mathcal{L}'

$$[a_0, b_0] = [a_0, b_1] = [L, b_1] = [b_0, b_1] = [b_0, L] = 0 \tag{106a}$$

$$[a_0, L] = \frac{1}{\nu - 2} b_1 \tag{106b}$$

for $\epsilon = 0$.

The proof is straightforward. In fact, under the assumption that $\nu \neq \dots, -3, -2, -1, \dots$, from (103b) we obtain $a_k = 0, [b_k, L] = 0$ ($k = 1, 2, 3, \dots$) and the commutation relation (105c). In addition, equation (104) entails the commutation relation (105d) and $b_2 = b_3 = \dots = 0$. Furthermore, equation (105b) can be determined by elaborating equation (105c) via the Jacobi identity applied to $[a_0, [b_0, L]]$. The commutation relations (106) emerge immediately.

A matrix representation of \mathcal{L} is

$$\begin{aligned} a_0 &= \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & b_0 &= \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ b_1 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & L &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2-\nu} \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{107}$$

Then, equations (80) take the form

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_x = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & u_t & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \tag{108a}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_t = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ (1 + \frac{u}{\gamma})^{\nu-1} & \frac{\nu-1}{\gamma} u_x (1 + \frac{u}{\gamma})^{\nu-2} & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \tag{108b}$$

from which $y_1 = \lambda_1 e^{\epsilon t}$, $y_2 = (\lambda_0 - \lambda_1 x) e^{\epsilon t}$, and

$$y_{3\zeta} = -\frac{1}{\lambda_1} e^{\epsilon t} u_t \zeta \tag{109a}$$

$$y_{3t} = -\lambda_1 e^{\epsilon t} \zeta^2 \frac{\partial}{\partial \zeta} \frac{(1 + \frac{u}{\gamma})^{\nu-1}}{\zeta} \tag{109b}$$

where $\zeta = \lambda_0 - \lambda_1 x$, and λ_0, λ_1 are constants of integration. Here y_3 can be interpreted as a potential variable.

Equations (108a) and (108b) allow us to find, in theory, special explicit solutions to equation (1). For example, let us assume that $u_t = \beta(t) \zeta^{\frac{2}{\nu-2}}$. Then, equations (109a) and (109b) give rise to the differential equation for $\beta(t)$

$$\beta = \gamma b^{\frac{1}{\nu-1}} \frac{d}{dt} (\beta + \epsilon \beta)^{\frac{1}{\nu-1}} \tag{110}$$

where $b = \frac{(\nu-2)^2}{2\nu(\nu-1)} \lambda_1^2$.

Equation (110) can be written as

$$\ddot{V} + \epsilon \dot{V} = k V^{\nu-1} \tag{111}$$

($k = \frac{b^{-\frac{1}{\nu-1}}}{\gamma}$) through the position

$$\dot{\beta} + \epsilon \beta = V^{\nu-1}. \tag{112}$$

For $\epsilon = 0$, we may use for the algebra \mathcal{L}' the same representation (107), where now b_0 is the null matrix. In this case equation (111) coincides with equation (40), which leads to explicit solutions of the original equation (5) for special values of the parameter γ (see section 3). We remark that via the change of variable $V = e^{\frac{2}{2-\gamma}W}$ and by a suitable choice of the free constant k , equation (111) becomes equation (38), which arises from the symmetry operator X_4^0 in the context of the Lie group theory. This fact suggests the existence of a possible link between the prolongation method and the symmetry approach. However, our result is indicative only, and many aspects of the problem remain to be elucidated.

4. Comments

We have dealt with a nonlinear field equation arising as the continuous limit of a lattice model containing many cases of physical significance (the harmonic, Toda, Fermi–Pasta–Ulam, Coulomb-like lattices and others). This equation, which is new, to the best of our knowledge, can be considered as a paradigm for the application of different analytical procedures. The addition of a perturbative friction-like term has allowed us to check the limits of validity of these methods. These are the symmetry and approximate-symmetry approach, and the prolongation technique. The joint use of these methods leads to the discovery of some interesting properties of equation (1), which have been expounded in the introduction.

The spirit of this paper is both of a methodological and speculative character. The results obtained are a challenge to apply the same strategy to investigate a three-dimensional version of equation (1). This purpose might be important in nonlinear field theories based on the deformation of the algebra used in the study of dispersionless field equations [21].

References

- [1] Leo M, Leo R A and Soliani G 1977 *Phys. Lett.* **60A** 283
- [2] Toda M 1981 *Theory of Nonlinear Lattices (Solid-State Science 20)* (Berlin: Springer)
- [3] Fermi E, Pasta J and Ulam S 1965 *Los Alamos Report LA-1940 (Collected Papers of Enrico Fermi)* vol 2, ed E Segre (Chicago, IL: University of Chicago Press) p 978
- [4] See, for example, Saveliev M V 1992 *Theor. Math. Fiz.* **92** 457 and references therein
- [5] Finley J D and Plebanski J F 1979 *J. Math. Phys.* **20** 1939
- [6] Boyer C P and Finley J D 1982 *J. Math. Phys.* **23** 1126
- [7] Dubrovin B and Novikov S P 1983 *Sov. Math. Dokl.* **27** 665
Tsarev S P 1985 *Sov. Math. Dokl.* **31** 488
Gümral H and Nutku Y 1990 *J. Math. Phys.* **31** 2606
Ward R S 1990 *Class. Quantum Grav.* **7** L95
Fairlie D B and Strachan I A B 1996 *Physica* **1D**
- [8] Olver P J 1986 *Applications of Lie Groups to Differential Equations* (New York: Springer)
- [9] Baikov V A, Gazizov R A and Ibragimov N H 1988 *Mat. Sbornik* **136** 435 (Engl. transl. 1989 *Math. USSR Sbornik* **64** 427)
- [10] Wahlquist H D and Estabrook F B 1975 *J. Math. Phys.* **16** 1
- [11] Alfinito E, Profilo G and Soliani G 1997 *J. Phys. A: Math. Gen.* **30** 1527
Przanowski M 1990 *J. Math. Phys.* **31** 300
Przanowski M 1990 *J. Math. Phys.* **31** 653
- [12] Lebrun C 1991 *J. Diff. Geom.* **34** 223
- [13] Barone A, Penna V and Sodano P 1993 *Ann. Phys., NY* **225** 212
- [14] Abramowitz M and Stegun I (ed) 1975 *Handbook of Mathematical Functions* (New York: Dover)
- [15] Killingbeck J 1978 *Phys. Lett.* **67A** 13
- [16] Chhajlany S C and Letov D A 1991 *Phys. Rev. A* **44** 4725 and references therein
- [17] Hagedorn R and Rafelski J 1983 *Commun. Math. Phys.* **83** 563

- [18] Tricomi F G 1954 *Funzioni Ipergeometriche Confluenti* (Rome: Edizioni Cremonese)
- [19] Gradshteyn I S and Ryzhik I M 1965 *Table of Integrals Series and Products* (New York: Academic)
- [20] Nutku Y, Sheftel M B and Malykh A 1997 *Class. Quantum Grav.* **14** L59
- [21] Strachan I A B 1996 *J. Phys. A: Math. Gen.* **29** 6117